

A covariant formalism for Chern–Simons gravity

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2003 J. Phys. A: Math. Gen. 36 2589

(<http://iopscience.iop.org/0305-4470/36/10/318>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.96

The article was downloaded on 02/06/2010 at 11:28

Please note that [terms and conditions apply](#).

A covariant formalism for Chern–Simons gravity

Andrzej Borowiec¹, Marco Ferraris² and Mauro Francaviglia²

¹ Institute of Theoretical Physics, University of Wrocław, pl. Maksa Borna 9, 50–204 Wrocław, Poland

² Dipartimento di Matematica, Università di Torino, Via C Alberto 10, 10123 Torino, Italy

Received 9 October 2002, in final form 28 January 2003

Published 26 February 2003

Online at stacks.iop.org/JPhysA/36/2589

Abstract

Chern–Simons type Lagrangians in $d = 3$ dimensions are analysed from the point of view of their covariance and globality. We use the transgression formula to find out a new fully covariant and global Lagrangian for Chern–Simons gravity: the price for establishing globality is hidden in a bimetric (or biconnection) structure. Such a formulation allows us to calculate from a global and simpler viewpoint the energy–momentum complex and the superpotential both for Yang–Mills and gravitational examples.

PACS numbers: 11.10.Ef, 11.10.Lm, 04.20.Fy

1. Introduction and preliminaries

It is well known that Einstein's gravity is trivial in dimension $d = 2$, since the curvature tensor reduces essentially to a scalar. Also in dimension $d = 3$ Einstein theory of gravitation is somehow trivial, since the Riemann tensor reduces essentially to the Einstein tensor. Because of this a generalization of the standard Hilbert Lagrangian was suggested in $d = 3$, by introducing, in full analogy with gauge theories, additional terms of the Chern–Simons type [1]. In a previous paper of ours [2] we have tackled the specific problem of conservation laws for Chern–Simons type Lagrangians, both in the Yang–Mills and in the gravitational case. In particular, we have calculated the relevant energy–momentum complex and the superpotential for Chern–Simons gravity in dimension $d = 3$ (see also [3] in this context). Another technique to compute superpotentials for Chern–Simons gauge theory which is based on the so-called cascade equation formalism [4] has been recently proposed in [5].

Let us recall that Chern–Simons Lagrangians for gravity are non-covariant (and non-global in general) due to the presence of cubic terms in the connection and a non-covariant coupling of curvature and connection, although field equations turn out to be global and covariant. Because of this and for the sake of simplicity, our result of [2] was obtained in a non-covariant framework as well as by assuming explicitly that spacetime has a trivial topology, i.e. assuming it to be globally diffeomorphic to an open subset of \mathbb{R}^3 . The aim of the present paper is thence to provide a 'covariantized' version of our previous calculations,

by relying on the ‘background connection method’, a covariantization procedure which has revealed itself to be rather useful in the case of first-order gravity [6–8]. In fact, the present paper is based on and should be considered as a direct continuation of [2]. In particular, we shall use the methods for computing currents and superpotentials as presented therein.

It is known that *natural*, i.e. generally covariant, Lagrangians lead to covariant Euler–Lagrange equations of motion. The inverse statement is, in general, not true. For example, as was mentioned before, this holds for the following non-global (in general) and non-invariant metric Lagrangian in dimension $d = 3$:

$$L_{CSG} = \frac{1}{2}\varepsilon^{\mu\nu\rho} (R_{\beta\mu\nu}^{\alpha} \Gamma_{\alpha\rho}^{\beta} - \frac{2}{3}\Gamma_{\beta\mu}^{\alpha} \Gamma_{\sigma\nu}^{\beta} \Gamma_{\alpha\rho}^{\sigma}) \quad (1)$$

where $\Gamma_{\beta\mu}^{\alpha}$ and $R_{\beta\mu\nu}^{\alpha}$ are the Christoffel symbols and the Riemann curvature tensor of a metric $g_{\mu\nu}$, respectively; here $\alpha, \mu \dots = 1, 2, 3$. The Lagrangian (1) leads, when varied with respect to the metric, to the following global and covariant tensorial Euler–Lagrange equations

$$C^{\alpha\beta} \equiv 2\varepsilon^{\mu\nu(\alpha} R_{\mu;\nu}^{\beta)} = 0 \quad (2)$$

where $C^{\alpha\beta}$ is called the *York–Cotton tensor density* and semicolon denotes metric covariant derivative. This symmetric and traceless tensor density vanish if and only if $R_{\alpha\beta}$ is the Ricci curvature tensor of a locally conformally flat metric g (see [9, 10]). Here and above $\varepsilon^{\mu\nu\alpha}$ denotes the relevant skew-symmetric Levi-Civita tensor density. It has been shown in [11] that the Lagrangian (1) is the only obstruction to the equivariant inverse problem in $d = 3$. This Lagrangian is the gravitational counterpart of Chern–Simons Lagrangians of gauge theories [1, 2].

A similar situation occurs in fact for the case of Chern–Simons gauge Lagrangian

$$L_{CS} = \frac{1}{2}\varepsilon^{\mu\nu\rho} \text{tr} (\mathbf{F}_{\mu\nu} \mathbf{A}_{\rho} - \frac{2}{3} \mathbf{A}_{\mu} \mathbf{A}_{\nu} \mathbf{A}_{\rho}) \quad (3)$$

where \mathbf{A}_{μ} is a matrix-valued gauge potential, its curvature 2-form $\mathbf{F}_{\mu\nu} = \partial_{\mu} \mathbf{A}_{\nu} - \partial_{\nu} \mathbf{A}_{\mu} + [\mathbf{A}_{\mu}, \mathbf{A}_{\nu}]$ being the gauge field strength and tr denoting the trace operation for matrices (in any suitable matrix group). The Lagrangian (3) is not gauge covariant although the corresponding field equations $\mathbf{F} = 0$ are. For this reason such type of Lagrangians are sometimes called *quasi (or almost) invariant*.

2. Transgression formula and covariant Chern–Simons Lagrangians

Let G be any Lie group and let us denote by \mathfrak{g} the corresponding Lie algebra. For simplicity, we shall think of G as a matrix group and \mathfrak{g} as a matrix algebra with the commutator $[\cdot, \cdot]$ as a Lie bracket. Consider a principal G -bundle P over a manifold M (which, for the moment, is arbitrary) with a principal connection ω on P . Its curvature 2-form is defined by $\Omega = d\omega + \omega \wedge \omega$ and fulfils the Bianchi identities $D\Omega \equiv d\Omega + [\omega, \Omega] = 0$. Recall (see e.g. [12, 13]) that ω is a \mathfrak{g} -valued and G -equivariant 1-form that lives on the total space P and which is not defined on the base manifold M . Choosing a (local) section $e : M \rightarrow P$ of P we get via pull-back a (local) matrix-valued 1-form $\mathbf{A}^{(e)} \equiv e^* \omega$ which lives on an open domain $U \subseteq M$. This is the familiar gauge potential (or Yang–Mills gauge field). In local coordinates $\{x^{\mu}\}$ on M it reads as $\mathbf{A}^{(e)} = \mathbf{A}_{\mu}^{(e)} dx^{\mu}$. A change of the local section $e \mapsto e' = eu$, with $u \in C^{\infty}(U, G)$, implies a non-tensorial transformation law for the corresponding (local) gauge potentials $\mathbf{A}^{(e)} \mapsto \mathbf{A}^{(e')} = u^{-1} \mathbf{A}^{(e)} u + u^{-1} du$; u is also called a gauge transformation. The (local) Yang–Mills field strength 2-form $\mathbf{F}^{(e)} \equiv e^* \Omega$, however, undergoes a tensorial transformation rule $\mathbf{F}^{(e)} \mapsto \mathbf{F}^{(e')} = u^{-1} \mathbf{F}^{(e)} u$. Because of this Ω is called a *tensorial* 2-form (see e.g., example 5.2, p 76 in [12], for the correspondence between tensorial forms on P and vector-valued forms on the base M). In local coordinates we shall write $\mathbf{F}^{(e)} = \frac{1}{2} \mathbf{F}_{\mu\nu}^{(e)} dx^{\mu} \wedge dx^{\nu}$.

In contrast, ω is a *non-tensorial* (but vertical) 1-form. By an abuse of notation from now on we shall drop all upper indications to the section e .

For any two principal connection 1-forms ω and $\bar{\omega}$ on P , Chern and Simons [9, 10] have established the famous *transgression formula*

$$\text{tr}(\Omega \wedge \Omega) - \text{tr}(\bar{\Omega} \wedge \bar{\Omega}) = d[Q_T(\omega, \bar{\omega})] \tag{4}$$

expressing the difference between two tensorial 4-forms. Here

$$Q_T(\omega, \bar{\omega}) \equiv \text{tr} \left(2\Omega \wedge \alpha - d\alpha \wedge \alpha - 2\omega \wedge \alpha \wedge \alpha + \frac{2}{3}\alpha \wedge \alpha \wedge \alpha \right) \tag{5}$$

denotes the so-called transgression 3-form (see, e.g., [13] p 348), with $\alpha = \omega - \bar{\omega}$ and $\bar{\Omega} = d\bar{\omega} + \bar{\omega} \wedge \bar{\omega}$. Note that $\text{tr}(\Omega \wedge \Omega)$ is a tensorial scalar-valued 4-form on P . Therefore, it uniquely determines the corresponding 4-form on the base manifold M , since $e^*(\text{tr}(\Omega \wedge \Omega)) = \frac{1}{4} \text{tr}(\mathbf{F}_{\mu\nu} \mathbf{F}_{\rho\sigma}) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$ does not depend on e . The transgression form (5) is an interesting and intriguing object in its own right. It can be easily re-expressed as

$$Q_T(\omega, \bar{\omega}) = \text{tr} \left(2\bar{\Omega} \wedge \alpha + \bar{D}\alpha \wedge \alpha + \frac{2}{3}\alpha \wedge \alpha \wedge \alpha \right) \tag{6}$$

where $\bar{D}\alpha = d\alpha + [\bar{\omega}, \alpha]$ denotes the covariant derivative of α with respect to the connection $\bar{\omega}$. Since α is tensorial, being the difference of two connections, the form $Q_T(\omega, \bar{\omega})$ is also a tensorial scalar-valued 3-form on P which uniquely determines the corresponding 3-form on the base manifold M .

Formula (4) expresses a well-known fact: although the Chern 4-form $\text{tr}(\Omega \wedge \Omega)$ itself depends on the connection, its cohomology class $[\text{tr}(\Omega \wedge \Omega)] \in H^4(M, \mathbb{R})$ in the de Rahm cohomology of M is connection independent since the difference $\text{tr}(\Omega \wedge \Omega) - \text{tr}(\bar{\Omega} \wedge \bar{\Omega})$ is exact. In more physical terms we can also say that the Chern form, when considered as a Lagrangian, whenever $\dim M = 4$, is variationally trivial since its variation

$$\delta \text{tr}(\Omega \wedge \Omega) = 2d \text{tr}(\delta\omega \wedge \Omega) \tag{7}$$

is a total divergence.

From now on we shall assume that the base manifold M is a 3-manifold. In this case $Q_T(\omega, \bar{\omega})$ is also closed since, of course, any 4-form on a 3-manifold vanishes identically. Because of this it determines a cohomology class $[Q_T(\omega, \bar{\omega})] \in H^3(M, \mathbb{R})$ which, in general, does not need to be trivial since $Q_T(\omega, \bar{\omega})$ need not be exact.

In particular, by replacing $\bar{\omega} = 0$ into (5) one immediately recognizes the well-known Chern–Simons 3-form:

$$P_T(\omega) = Q_T(\omega, 0) = \text{tr} \left(d\omega \wedge \omega + \frac{2}{3}\omega \wedge \omega \wedge \omega \right) \equiv \text{tr} \left(\Omega \wedge \omega - \frac{1}{3}\omega \wedge \omega \wedge \omega \right). \tag{8}$$

The Chern–Simons form (8) is also a closed, scalar-valued, but non-tensorial 3-form, which lives on the principal bundle P and not on the base manifold. Therefore, it determines a cohomology class $[P_T(\omega)] \in H^3(P, \mathbb{R})$ in the de Rahm cohomology of P , which, in general, may depend on the connection. To see this one can use arguments similar to those presented in [9] (lemma 3.10). For this purpose we calculate with a bit of algebra the following:

$$Q_T(\omega, \bar{\omega}) = P_T(\omega) - P_T(\bar{\omega}) - d \text{tr}(\omega \wedge \bar{\omega}). \tag{9}$$

Now it is clear that the element $[Q_T(\omega, \bar{\omega})] \in H^3(M, \mathbb{R})$ measures, in a certain sense, the difference between the cohomology classes $[P_T(\omega)]$ and $[P_T(\bar{\omega})]$. These classes are called *secondary characteristic classes* for a manifold with connection.

The local Lagrangians (3) can be obtained from (8) by pull-back along local sections e of P . If any global section exists, i.e. if P is a trivializable bundle³ one can use it to construct a

³ If the group G is simply connected then any principal G -bundle over a 3-manifold is trivializable [14].

global Lagrangian. In this case the corresponding action integral

$$\mathcal{A}_M(\omega) = \frac{1}{8\pi\kappa} \int_e P_T(\omega) \equiv \frac{1}{8\pi\kappa} \int_M e^*(P_T(\omega)) \equiv \frac{1}{8\pi\kappa} \int_M L_{CS}(\mathbf{A}) \quad (10)$$

is multivalued since its value depends on the section chosen [14–16]. In fact, Chern and Simons found that this dependence is up to a homology class of the section e , therefore it must have a non-dynamical character. After introducing an appropriate normalization constant κ it turns out that the actions corresponding to homologically non-equivalent sections differ by integer values (the so-called winding number). Alternatively, one can say that the action depends on the connection and takes its values in the quotient \mathbb{R}/\mathbb{Z} . In other words it produces a (secondary) characteristic number for a 3-manifold with connection (see [14] for an exhaustive discussion).

To resume, fixing any (global) section, the Chern–Simons form (8) pulls down to M and gives the Lagrangian (3). This non-invariant Lagrangian produces, however, invariant and geometrically simple equations of motion. Indeed, the variation of (8) gives rise to the following expression (see [2])

$$\delta P_T = 2 \operatorname{tr}(\delta\omega \wedge \Omega) + d \operatorname{tr}(\delta\omega \wedge \omega) \quad (11)$$

which of course yields $\Omega = 0$ as the equation of motion. Since $\delta\omega$ is tensorial, the Euler–Lagrange part is tensorial too and one realizes that the whole non-invariance has passed into the boundary term $\operatorname{tr}(\delta\omega \wedge \omega)$. This implies that the corresponding canonical Nöther currents and superpotentials are not tensorial (compare formulae (26), (27), (30) and (31) in [2]). It means that they are gauge (i.e., section) dependent or in other words they live on the total space of the bundle P .

Our main idea in the present paper is to use

$$L_T(\mathbf{A}, \bar{\mathbf{A}}) = e^*(Q_T(\omega \cdot \bar{\omega})) \quad (12)$$

as a Lagrangian 3-form on M . We stress again that L_T is a global and covariant object which lives on the base manifold M . This fact is independent of the topologies of P , G and M . However, the price one has to pay for this is the biconnection character of the Lagrangian (12). We shall analyse two cases: (i) both connections are dynamical; (ii) only ω is dynamical while $\bar{\omega}$ is a fixed background (non-dynamical) connection.

In terms of physically more relevant (but local) quantities $\omega = \mathbf{A}_\mu dx^\mu$, $\Omega = \frac{1}{2}\mathbf{F}_{\mu\nu} dx^\mu \wedge dx^\nu$ and $\alpha = \mathbf{B}_\mu dx^\mu \equiv (\mathbf{A}_\mu - \bar{\mathbf{A}}_\mu) dx^\mu$, according to (6) and (9) one has

$$\begin{aligned} L_T(\mathbf{A}, \bar{\mathbf{A}}) &= \varepsilon^{\mu\nu\rho} \operatorname{tr}(\bar{\mathbf{F}}_{\mu\nu}\mathbf{B}_\rho + (\bar{D}_\mu\mathbf{B}_\nu)\mathbf{B}_\rho + \frac{2}{3}\mathbf{B}_\mu\mathbf{B}_\nu\mathbf{B}_\rho) \\ &\equiv \frac{1}{2}\varepsilon^{\mu\nu\rho} \operatorname{tr}(\mathbf{F}_{\mu\nu}\mathbf{A}_\rho - \frac{2}{3}\mathbf{A}_\mu\mathbf{A}_\nu\mathbf{A}_\rho - \bar{\mathbf{F}}_{\mu\nu}\bar{\mathbf{A}}_\rho + \frac{2}{3}\bar{\mathbf{A}}_\mu\bar{\mathbf{A}}_\nu\bar{\mathbf{A}}_\rho) \\ &\quad - \partial_\mu[\varepsilon^{\mu\nu\rho} \operatorname{tr}(\mathbf{A}_\nu\bar{\mathbf{A}}_\rho)] \end{aligned} \quad (13)$$

where $dx^\mu \wedge dx^\nu \wedge dx^\rho = \varepsilon^{\mu\nu\rho} dx^1 \wedge dx^2 \wedge dx^3$ has been used and $\bar{D}_\mu \equiv \partial_\mu + [\bar{\mathbf{A}}_\mu, \cdot]$ denotes the directional covariant derivative with respect to the connection $\bar{\omega}$. Now the Lagrangian L_T is represented by a scalar density of weight one rather than a 3-form (see [2]).

The variation of (12) is easily calculated from (9) and (11); we get:

$$\delta Q_T = 2 \operatorname{tr}(\delta\omega \wedge \Omega) - 2 \operatorname{tr}(\delta\bar{\omega} \wedge \bar{\Omega}) + d \operatorname{tr}((\delta\omega + \delta\bar{\omega}) \wedge \alpha). \quad (14)$$

Accordingly, (14) reads now as

$$\delta L_T(\mathbf{A}, \bar{\mathbf{A}}) = \varepsilon^{\mu\nu\rho} \operatorname{tr}(\mathbf{F}_{\mu\nu}\delta\mathbf{A}_\rho - \bar{\mathbf{F}}_{\mu\nu}\delta\bar{\mathbf{A}}_\rho) - \partial_\mu[\varepsilon^{\mu\nu\rho} \operatorname{tr}(\mathbf{B}_\nu(\delta\mathbf{A}_\rho + \delta\bar{\mathbf{A}}_\rho))]. \quad (15)$$

An infinitesimal pure gauge transformation is given by means of a matrix-valued function (0-form) χ . One has

$$\delta_\chi \mathbf{A}_\mu = D_\mu \chi \quad \delta_\chi \bar{\mathbf{A}}_\mu = \bar{D}_\mu \chi \quad \text{and} \quad \delta_\chi L_T = 0 \quad (16)$$

i.e., L_T is a gauge scalar. With this in mind we are able to calculate the canonical Nöther current associated with a gauge symmetry as

$$J_T^\mu(\chi) = \varepsilon^{\mu\nu\rho} \text{tr}[\mathbf{B}_\nu(D_\rho \chi + \bar{D}_\rho \chi)] \quad (17)$$

(compare with the calculations given in [2]). This quantity is weakly conserved. Due to the second Nöther theorem, it decomposes into the so-called reduced current (which vanishes on shell) and the *superpotential* [2, 4–6, 17, 18]. The superpotential is known to represent that part of a current which is identically conserved, does not vanish on shell and which is enough for the computation of conserved quantities (such as charges, masses and so on). In this case one gets explicitly (see [2])

$$\begin{aligned} J_T^\mu(\chi) &= 2\partial_\nu[\varepsilon^{\mu\nu\rho} \text{tr}(\mathbf{B}_\nu \chi)] + \varepsilon^{\mu\nu\rho} \text{tr}(\chi(D_\rho \mathbf{B}_\nu + \bar{D}_\rho \mathbf{B}_\nu)) \\ &= \partial_\rho U_T^{\mu\rho} + \varepsilon^{\mu\nu\rho} \text{tr}(\chi \mathbf{F}_{\nu\rho} - \chi \bar{\mathbf{F}}_{\nu\rho}) \end{aligned} \quad (18)$$

where the superpotential $U_T^{\mu\rho} = -U_T^{\rho\mu}$ takes the very simple form

$$U_T^{\mu\rho}(\chi) = 2\varepsilon^{\mu\nu\rho} \text{tr}(\mathbf{B}_\nu \chi). \quad (19)$$

The above decomposition can be easily justified by using the identity $(D + \bar{D})\alpha = 2(\Omega - \bar{\Omega})$ and by the following formula:

$$\text{tr}(D\chi \wedge \alpha) = \text{tr}(D(\chi\alpha)) - \text{tr}(\chi D\alpha) = d \text{tr}(\chi\alpha) - \text{tr}(\chi D\alpha) \quad (20)$$

which holds true since the trace vanishes on commutators.

A similar analysis can be performed for the diffeomorphism invariance of L_T . Any vector field $\xi = \xi^\mu \partial_\mu$ on M is just an infinitesimal diffeomorphism. Under diffeomorphisms the gauge potentials \mathbf{A}_μ and $\bar{\mathbf{A}}_\mu$ behave (at least locally; see the discussion below) as 1-forms and L_T as a scalar density of weight one. An infinitesimal diffeomorphism transformation acts on any (natural) geometric object over M by means of the Lie derivative \mathcal{L}_ξ . In particular,

$$\delta_\xi \mathbf{A}_\mu \equiv \mathcal{L}_\xi \mathbf{A}_\mu = \xi^\alpha \partial_\alpha \mathbf{A}_\mu + \mathbf{A}_\alpha \partial_\mu \xi^\alpha \quad \delta_\xi L_T \equiv \mathcal{L}_\xi L_T = \partial_\alpha (\xi^\alpha L_T) \quad (21)$$

and similarly for $\delta_\xi \bar{\mathbf{A}}_\mu \equiv \mathcal{L}_\xi \bar{\mathbf{A}}_\mu$. This leads to the following expression for the Nöther current

$$J_T^\mu(\xi) = \xi^\mu L_T + \varepsilon^{\mu\nu\rho} \text{tr}[\mathbf{B}_\nu \partial_\alpha (\mathbf{A}_\rho + \bar{\mathbf{A}}_\rho)] \xi^\alpha + \varepsilon^{\mu\nu\rho} \text{tr}[\mathbf{B}_\nu (\mathbf{A}_\alpha + \bar{\mathbf{A}}_\alpha)] \partial_\rho \xi^\alpha \quad (22)$$

and

$$U_T^{\mu\rho}(\xi) = \varepsilon^{\mu\nu\rho} \text{tr}[\mathbf{B}_\nu (\mathbf{A}_\xi + \bar{\mathbf{A}}_\xi)] \quad (23)$$

for the corresponding superpotential (compare with formulae (31) and (32) in [2]). Here for simplicity we introduced the shortcut $\mathbf{A}_\xi \equiv \mathbf{A}_\alpha \xi^\alpha$.

Note that the expressions (22) and (23) are not gauge covariant since they do contain gauge non-covariant terms such as \mathbf{A}_ξ and $\bar{\mathbf{A}}_\xi$ as well as terms involving the partial derivatives. A similar situation is also known in Yang–Mills theory, since the formal Lie derivative (21) does not fill the matrix degrees of freedom. Strictly speaking, the group $\text{Diff}(M)$ of all diffeomorphisms of M is not valid as a global invariance group for the theory. The most general symmetry group is the group $\text{Aut}_G(P)$ which consists of all G -invariant bundle automorphisms $\Phi : P \rightarrow P$, i.e. the so-called principal G -automorphisms of P . The group $\text{Gauge}(P)$ of all pure gauge transformations is in a natural way a subgroup of $\text{Aut}_G(P)$, while $\text{Diff}(M)$ is not. One has instead a surjective group homomorphism from $\text{Aut}_G(P)$ onto $\text{Diff}(M)$. The kernel of this homomorphism is of course $\text{Gauge}(P)$. It is clear that an

infinitesimal automorphism ϕ of the principal bundle P is generated by the corresponding G -invariant projectable vector field on P and it can be represented (at least locally) as a pair $\phi = (\xi, \chi)$ with a vector field ξ uniquely defined [19] (see also discussion in [3]). Fixing some background principal connection $\omega_o \equiv \mathbf{a}_\mu dx^\mu$ on P and choosing $\chi = -\mathbf{a}_\xi$ we may use the formula

$$\delta_{(\xi,a)} \mathbf{A}_\mu \equiv \mathcal{L}_\xi \mathbf{A}_\mu - D_\mu \mathbf{a}_\xi = \xi^\alpha \mathbf{F}_{\mu\alpha} + D_\mu (\mathbf{A}_\xi - \mathbf{a}_\xi) \quad (24)$$

in order to lift the vector field ξ on M into the corresponding G -invariant projectable vector field on P . Such a lifting is not canonical, being background dependent, but it is global. Moreover, ω_o is flat if and only if the corresponding lift is a Lie algebra map. This remark generalizes the so-called improved diffeomorphism technique presented in [20–22].

We can conclude this part by saying that the diffeomorphism invariance of a Yang–Mills type Lagrangian is encoded into the invariance with respect to the lifted diffeomorphisms, i.e. the corresponding principal automorphisms of P . In our case, the total superpotential related to the diffeomorphism invariance of the Lagrangian (6) has the form

$$U_T^{\mu\rho}(\xi, \mathbf{a}) = \varepsilon^{\mu\nu\rho} \text{tr}[\mathbf{B}_\nu (\mathbf{A}_\xi + \bar{\mathbf{A}}_\xi - 2\mathbf{a}_\xi)] \quad (25)$$

which is covariant but depends on the background. Finally, choosing $\mathbf{a} = \bar{\mathbf{A}}$ we find

$$U_T^{\mu\rho}(\xi, \bar{\mathbf{A}}) = \varepsilon^{\mu\nu\rho} \text{tr}(\mathbf{B}_\nu \mathbf{B}_\xi) \quad (26)$$

which is fully covariant and background independent, provided $\bar{\mathbf{A}}$ is a dynamical connection. Note, that for a diagonal solution $\mathbf{A} = \bar{\mathbf{A}}$ all expressions for the superpotential automatically vanish, while the limit $\bar{\mathbf{A}} \rightarrow 0$ reproduces the results previously given in [2].

Alternatively, let us now assume that the connection $\bar{\omega}$ is a fixed background (non-dynamical) connection. Thus $\delta\bar{\omega} = 0$ in (14). The theory in this case has only one dynamical field but the class of symmetries is more restrictive: the gauge transformations have to keep the background unchanged, i.e. $\delta_\chi \bar{\mathbf{A}}_\mu \equiv \bar{D}_\mu \chi = 0$. This implies $D_\mu \chi = [\mathbf{B}_\mu, \chi]$ and

$$J_T^\mu(\chi) = 2\varepsilon^{\mu\nu\rho} \text{tr}(\mathbf{B}_\nu \mathbf{B}_\rho \chi) \equiv \varepsilon^{\mu\nu\rho} \text{tr}([\mathbf{B}_\nu, \mathbf{B}_\rho] \chi). \quad (27)$$

This last expression vanishes identically in the case of an Abelian gauge group.

Recently, the so-called mixed Chern–Simons term based on two independent $U(1)$ -gauge fields, one of electromagnetic origin and the other statistical, has been successfully applied in two-dimensional superconductivity (see [23] and references quoted therein)⁴.

3. Bi-metric Chern–Simons gravity

A particularly interesting situation appears when P is the bundle of linear frames LM , so that the group G is the general linear group $GL(3, \mathbb{R})$. Linear connections on M are principal connections in LM .

In this case ω is a $\mathfrak{gl}(3, \mathbb{R})$ -valued 1-form on the bundle LM representing a linear connection on M and Ω is its Riemann curvature 2-form. We can use a coordinate section (gauge) $\{\partial_\mu\}$ to write down $\omega = \Gamma_\mu dx^\mu$ and $\Omega = \frac{1}{2} \mathbf{R}_{\mu\nu} dx^\mu \wedge dx^\nu$, where $\Gamma_\mu \equiv \Gamma_{\beta\mu}^\alpha$ and $\mathbf{R}_{\mu\nu} \equiv R_{\beta\mu\nu}^\alpha$ are the standard local expressions for the connection coefficients and its Riemann curvature tensor represented now as 3×3 matrices. Alternatively, we can also use a local (but not necessarily coordinate) section $\{E_i = E_i^\mu \partial_\mu\}$, the so-called dreibein. In this case, the matrix indices $\Gamma_\mu \equiv \Gamma_{j\mu}^i$ and $\mathbf{R}_{\mu\nu} \equiv R_{j\mu\nu}^i$, the so-called world indices, are inherited from the dreibein $\{E_i\}$.

⁴ This comment is due to Ashoke Das.

The Chern–Simons 3-form (8) lives then on the bundle of linear frames LM and the (local) Lagrangian (1) can be obtained from (8) by pull-back along a coordinate section $\{\partial_\mu\}$ of LM . Having chosen a coordinate atlas on the base manifold, with any coordinate neighbourhood one can associate such a local Lagrangian. On the intersection of two neighbourhoods both Lagrangians differ by a total derivative. This defines a 0-cochain of local Lagrangians in the sense of Čech cohomology. Conservation laws for this type of non-global Lagrangian will be investigated in detail in [24]. If the manifold M is parallelizable (i.e. LM is a trivial bundle, which is always the case for a compact, oriented 3-manifold), one can also use a global (but probably no longer coordinate) dreibein to obtain a global but not invariant Lagrangian.

Assuming that the linear connection Γ is the Levi-Civita connection of some metric g on M :

$$\Gamma_{\beta\mu}^\alpha = \frac{1}{2}g^{\alpha\sigma}(\partial_\beta g_{\mu\sigma} + \partial_\mu g_{\sigma\beta} - \partial_\sigma g_{\beta\mu}) \tag{28}$$

i.e. considering g instead of Γ as the dynamical variable, we thus obtain Chern–Simons gravity theory. The corresponding action (10) is metric dependent and it produces the secondary invariant of a Riemannian manifold (M, g) (see [9, 10]).

The transgression form (6) gives then a new global and bimetric Lagrangian density for Chern–Simons gravity. The Lagrangian (13) now takes the form

$$\begin{aligned} L_{TG}(g, \bar{g}) &= \varepsilon^{\mu\nu\rho} \operatorname{tr}(\bar{\mathbf{R}}_{\mu\nu}\mathbf{N}_\rho + (\bar{\nabla}_\mu\mathbf{N}_\nu)\mathbf{N}_\rho + \frac{2}{3}\mathbf{N}_\mu\mathbf{N}_\nu\mathbf{N}_\rho) \\ &\equiv \frac{1}{2}\varepsilon^{\mu\nu\rho} \operatorname{tr}(\mathbf{R}_{\mu\nu}\Gamma_\rho - \frac{2}{3}\Gamma_\mu\Gamma_\nu\Gamma_\rho - \bar{\mathbf{R}}_{\mu\nu}\bar{\Gamma}_\rho + \frac{2}{3}\bar{\Gamma}_\mu\bar{\Gamma}_\nu\bar{\Gamma}_\rho) - \partial_\mu[\varepsilon^{\mu\nu\rho} \operatorname{tr}(\Gamma_\nu\bar{\Gamma}_\rho)] \end{aligned} \tag{29}$$

where $\mathbf{N}_\mu \equiv \Gamma_\mu - \bar{\Gamma}_\mu$ and $\bar{\nabla}_\mu \equiv \partial_\mu + [\bar{\Gamma}_\mu, \cdot]$ denote the Levi-Civita covariant derivative with respect to the metric \bar{g} . Again, since the difference of two connections is a tensorial 1-form $N^i_{j\mu} dx^\mu$ one plays exclusively with tensorial objects. Therefore, there is no need to distinguish between the world and the local indices. Accordingly, the Lagrangian density (29) is a global and dreibein independent 3-form on M . It is even fully covariant (i.e. natural) if one considers both metrics (g, \bar{g}) as dynamical fields. Now, it is well justified to use the local expression

$$\begin{aligned} L_{TG}(g, \bar{g}) &= \varepsilon^{\mu\nu\rho} (\bar{R}_{\beta\mu\nu}^\alpha N_{\alpha\rho}^\beta + (\bar{\nabla}_\mu N_{\beta\nu}^\alpha) N_{\alpha\rho}^\beta + \frac{2}{3}N_{\beta\mu}^\alpha N_{\sigma\nu}^\beta N_{\alpha\rho}^\sigma) \\ &\equiv \frac{1}{2}\varepsilon^{\mu\nu\rho} (R_{\beta\mu\nu}^\alpha \Gamma_{\alpha\rho}^\beta - \frac{2}{3}\Gamma_{\beta\mu}^\alpha \Gamma_{\sigma\nu}^\beta \Gamma_{\alpha\rho}^\sigma - \bar{R}_{\beta\mu\nu}^\alpha \bar{\Gamma}_{\alpha\rho}^\beta + \frac{2}{3}\bar{\Gamma}_{\beta\mu}^\alpha \bar{\Gamma}_{\sigma\nu}^\beta \bar{\Gamma}_{\alpha\rho}^\sigma) \\ &\quad - \partial_\mu [\varepsilon^{\mu\nu\rho} \Gamma_{\beta\nu}^\alpha \bar{\Gamma}_{\alpha\rho}^\beta] \end{aligned} \tag{30}$$

for the corresponding global 3-form on M . Variation of (30) with respect to the connections $(\Gamma, \bar{\Gamma})$ yields (compare with (15)):

$$\delta L_{TG} = \varepsilon^{\mu\nu\rho} (R_{\beta\mu\nu}^\alpha \delta\Gamma_{\alpha\rho}^\beta - \bar{R}_{\beta\mu\nu}^\alpha \delta\bar{\Gamma}_{\alpha\rho}^\beta) - \partial_\mu [\varepsilon^{\mu\nu\rho} N_{\beta\nu}^\alpha (\delta\Gamma_{\alpha\rho}^\beta + \delta\bar{\Gamma}_{\alpha\rho}^\beta)]. \tag{31}$$

In fact, the Lagrangian (30) and its variation (31) can be alternatively analysed from a first-order (in the style of Palatini) point of view, i.e. having just two linear connections $(\Gamma, \bar{\Gamma})$ as dynamical variables⁵. As a symmetry transformation consider then a one-parameter group of diffeomorphisms generated by the vector field $\xi = \xi^\alpha \partial_\alpha$. The Lie derivative of an arbitrary (non-symmetric) linear connection Γ reads (see e.g. [25, 26])

$$\delta_\xi \Gamma_\rho \equiv \mathcal{L}_\xi \Gamma_{\alpha\rho}^\beta = \xi^\sigma R_{\alpha\sigma\rho}^\beta + \nabla_\rho \nabla_\alpha^* \xi^\beta \tag{32}$$

where $\nabla_\alpha^* \xi^\beta = \partial_\alpha \xi^\beta + \Gamma_{\sigma\alpha}^\beta \xi^\sigma$ (remember that $\nabla_\alpha \xi^\beta = \partial_\alpha \xi^\beta + \Gamma_{\alpha\sigma}^\beta \xi^\sigma$). It defines a canonical natural lift from any vector field on M to the corresponding invariant projectable vector field

⁵ Of course, the bimetric and biconnection approaches are not equivalent since they lead to non-equivalent equations of motion.

on an appropriate bundle of geometric objects over M . In other words, the difference between this case and the general one discussed in the previous section is that the Lie transport now provides a canonical (i.e. background independent) embedding of $\text{Diff}(M)$ into the group of principal automorphisms of LM with a gauge part represented by $\chi = \nabla^*\xi$. Applying formula (23) to the present case one might be tempted to write

$$U_T^{\mu\rho}(\xi) = \varepsilon^{\mu\nu\rho} N_{\beta\nu}^\alpha (\nabla_\alpha^* + \bar{\nabla}_\alpha^*) \xi^\beta \tag{33}$$

for the corresponding superpotential. This is wrong since variation (32) is a second-order differential operator in ξ (see [2, 6, 7]).

It is now convenient to assume that both connections are symmetric (i.e., torsion free) linear connections on M . Thus following the same steps as for computations of formula (60) in [2] but this time in a covariant manner, i.e. having replaced $\Gamma_{\beta\mu}^\alpha$ by $N_{\beta\mu}^\alpha$ and the partial derivatives ∂_μ by the covariant ones ∇_μ or $\bar{\nabla}_\mu$, respectively, one gets

$$U_T^{\mu\rho}(\xi) = \frac{1}{6}\varepsilon^{\mu\nu\rho}[(\nabla_\sigma + \bar{\nabla}_\sigma)(3N_{\alpha\nu}^\sigma - \delta_\nu^\sigma N_{\alpha\beta}^\beta)]\xi^\alpha - \frac{1}{3}\varepsilon^{\mu\nu\rho}(3N_{\alpha\nu}^\sigma - \delta_\nu^\sigma N_{\alpha\beta}^\beta)(\nabla_\sigma + \bar{\nabla}_\sigma)\xi^\alpha. \tag{34}$$

Coming back to the purely metric formalism we wish to perform the variation of (30) with respect to the metrics (g, \bar{g}) . For this reason one has to replace $\delta\Gamma_\rho$ in the first term of (31) by means of the ‘Palatini formula’

$$\delta\Gamma_{\beta\rho}^\alpha = \frac{1}{2}g^{\alpha\sigma}(\nabla_\beta\delta g_{\rho\sigma} + \nabla_\rho\delta g_{\sigma\beta} - \nabla_\sigma\delta g_{\beta\rho}) \tag{35}$$

and the same for $\delta\bar{\Gamma}_\rho$. Accordingly, after some computation (see also [2]) the bimetric first variational formula reads now as

$$\delta L_{TG} = \bar{C}^{\alpha\rho}\delta\bar{g}_{\alpha\rho} - C^{\alpha\rho}\delta g_{\alpha\rho} + \partial_\mu[\varepsilon^{\mu\nu\rho}(2R_\nu^\alpha\delta g_{\alpha\rho} - 2\bar{R}_\nu^\alpha\delta\bar{g}_{\alpha\rho} - N_{\beta\nu}^\alpha(\delta\Gamma_{\alpha\rho}^\beta + \delta\bar{\Gamma}_{\alpha\rho}^\beta))] \tag{36}$$

where the York–Cotton tensor density $C^{\alpha\rho}$ (resp. $\bar{C}^{\alpha\rho}$) is given by (2). The Euler–Lagrange field equations are $C^{\alpha\rho} = \bar{C}^{\alpha\rho} = 0$. We recall that the York–Cotton tensor density is symmetric, traceless, divergence free and it vanishes if and only if the corresponding metric is conformally flat.

Again, as a symmetry transformation let us consider a flow of diffeomorphisms generated by the vector field $\xi = \xi^\alpha\partial_\alpha$. In this case, the Lie derivative operators

$$\delta_\xi g \equiv \mathcal{L}_\xi g_{\alpha\rho} = \nabla_\alpha\xi_\rho + \nabla_\rho\xi_\alpha \tag{37}$$

and

$$\delta_\xi \Gamma_\rho \equiv \mathcal{L}_\xi \Gamma_{\alpha\rho}^\beta = \xi^\sigma R_{\alpha\sigma\rho}^\beta + \nabla_\rho\nabla_\alpha\xi^\beta \tag{38}$$

represent the infinitesimal variations.

Consequently, the formulae (36), (37) and (38) allow us to calculate the canonical energy–momentum complex and superpotential in both covariant (bimetric) and background connection ($\delta\bar{\Gamma} \equiv 0$) formalisms. To this end we make use of the computations already performed in [2]. Only terms under the divergence in (36) will contribute to the superpotential. We see that the first two terms correspond to formula (56) in [2]. Therefore, combining with (34) we arrive at the following expression:

$$U_{TG}^{\mu\rho}(\xi) = \varepsilon^{\mu\nu\rho}[(3R_{\nu\alpha} - Rg_{\nu\alpha} - 3\bar{R}_{\nu\alpha} + \bar{R}\bar{g}_{\nu\alpha})\xi^\alpha + \frac{1}{6}\varepsilon^{\mu\nu\rho}[(\nabla_\sigma + \bar{\nabla}_\sigma)(3N_{\alpha\nu}^\sigma - \delta_\nu^\sigma N_{\alpha\beta}^\beta)]\xi^\alpha - \frac{1}{3}\varepsilon^{\mu\nu\rho}(3N_{\alpha\nu}^\sigma - \delta_\nu^\sigma N_{\alpha\beta}^\beta)(\nabla_\sigma + \bar{\nabla}_\sigma)\xi^\alpha. \tag{39}$$

As a concrete example one can consider a solution (g, \bar{g}) consisting of a flat metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ while $g_{\mu\nu} = \exp(2\phi)\eta_{\mu\nu}$ being conformal to η with a conformal factor ϕ . Having chosen $\xi^\alpha = \eta^{\alpha\beta}\phi_{,\beta}$ one calculates

$$U = dF + F d\phi \tag{40}$$

as a 1-form, where $F = -\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$. In particular, for $\phi = r \equiv \sqrt{\eta_{\alpha\beta}x^\alpha x^\beta}$ we obtain $U = -dr$, i.e.

$$U^{\mu\rho} = -\varepsilon^{\mu\nu\rho}\frac{x_\nu}{r}. \quad (41)$$

4. Conclusions

We have considered the Chern–Simons type models in three dimensions. Exploiting the Chern–Simons transgression 3-form enables us to find a new global Lagrangian density which unlike, the local Chern–Simons Lagrangian is generally covariant. However, in this approach the covariant Lagrangian has a biconnection character and the corresponding theory loses some of its topological properties. Particularly, the action functional becomes insensitive to the topology of underlying 3-manifolds. The formalism has been used for the calculation of conserved Nöther currents and their identically conserved parts—superpotentials. Two special cases are of particular interest: the case of two connections being dynamical and the case when one of the connections is given as a fixed background, while the other one is only dynamical. Finally, the Chern–Simons gravity has been treated in a similar way. In this sense, the present paper generalizes the results of our previous paper [2] obtained for non-covariant Chern–Simons Lagrangians (see also [3]). Recently, this covariant formalism has been successfully applied to explicit numerical calculations of conserved quantities for BTZ black-hole solutions in AdS_3 Chern–Simons gravity of the Witten type [27].

Acknowledgments

One of the authors (AB) would like to thank the INdAM–GNFM for providing financial support and the Department of Mathematics of Torino University for the hospitality during his stay in Italy. The authors are grateful to Marco Godina and Marcella Palese for fruitful discussions. Some comments by a referee are also acknowledged.

References

- [1] Deser S, Jackiw R and Templeton S 1982 *Ann. Phys., NY* **140** 372
- [2] Borowiec A, Ferraris M and Francaviglia M 1998 *J. Phys. A: Math. Gen.* **31** 8823 (*Preprint hep-th/9801126*)
- [3] Sardanashvili G 2002 Energy–momentum conservation laws in gauge theory with broken symmetry *Preprint hep-th/0203275*
- [4] Julia B and Silva S 1998 *Class. Quantum Grav.* **15** 2173
- [5] Silva S 1999 *Nucl. Phys. B* **558** 391
- [6] Ferraris M and Francaviglia M 1990 *Gen. Rel. Grav.* **22** 965
- [7] Ferraris M and Francaviglia M 1992 *Class. Quantum Grav.* **9** S79
- [8] Fatibene L, Ferraris M, Francaviglia M and Raiteri M 2001 *J. Math. Phys.* **42** 1173
- [9] Chern S S and Simons J 1971 *Proc. Natl Acad. Sci. USA* **68** 791
- [10] Chern S S and Simons J 1974 *Ann. Math* **99** 48
- [11] Anderson I 1984 *Ann. Math.* **120** 329
- [12] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* (New York: Wiley)
- [13] Eguchi T, Gilkey P B and Hanson A J 1980 *Phys. Rep.* **66** 213
- [14] Freed D S 1995 *Adv. Math.* **113** 237
- [15] Oziewicz Z 1992 *Rep. Math. Phys.* **31** 85
- [16] Chamseddine A H and Fröhlich J 1992 *Commun. Math. Phys.* **147** 549
- [17] Bak D, Cangemi D and Jackiw R 1994 *Phys. Rev. D* **49** 5173
- [18] Sardanashvili G 1997 *Class. Quantum Grav.* **14** 1371
- [19] Bergmann P G and Flaherty E J Jr 1978 *J. Math. Phys.* **19** 212
- [20] Jackiw R 1978 *Phys. Rev. Lett.* **41** 1635

-
- [21] Bañados M, Garay L J and Henneaux M 1996 *Phys. Rev. D* **53** R593
 - [22] Bañados M, Henneaux M, Iannuzzo C and Viallet C M 1997 *Class. Quantum Grav.* **14** 2455
 - [23] Aitchison I J R and Fosco C D 1998 *Phys. Rev. D* **57** 1171
 - [24] Borowiec A, Ferraris M, Francaviglia M and Palese M 2001 Conservation laws for non-global Lagrangians *Quaderni Dip. Mat. Univ. Torino* submitted (Preprint math-ph/0301043)
 - [25] Yano K 1957 *The Theory of Lie Derivatives and Its Applications* (Amsterdam: North-Holland)
 - [26] Schouten J A 1954 *Ricci-Calculus* 2nd edn (Berlin: Springer)
 - [27] Allemandi G, Francaviglia M and Raiteri M 2003 Covariant charges in Chern–Simons AdS₃ gravity *Class. Quantum Grav.* **20** 483